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New Exact Solutions to the Combined KdV and mKdV Equation

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The modified mapping method is developed to obtain new exact solutions to the combined KdV and mKdV equation. The method is applicable to a large variety of nonlinear evolution equations, as long as odd- and even-order derivative terms do not coexist in the equation under consideration.

KEY WORDS: combined KdV and mKdV equation; modified mapping method; exact solution; elliptic equation.

1. INTRODUCTION

There are many methods for obtaining exact solutions of a nonlinear evolution equation, such as the inverse scattering transformation, the bilinear method, symmetry reductions, Bäcklund, and Darboux transformations. Recently, directly searching for exact solutions of the nonlinear evolution equations has become more and more attractive for their important role in understanding the nonlinear phenomena. Some of the important methods are tanh-function method (Malfliet, 1992; Parkes *et al.*, 1997), sech-function method (Duffy and Parkes, 1996) and Jacobi elliptic function method (Fu *et al.*, 2001; Liu *et al.* 2001; Parkes *et al.*, 2002). In an earlier paper (Peng, to appear), the mapping method is proposed for obtaining exact solutions to nonlinear evolution equations. An advantage of this method is that we can obtain the solitary wave solution, the periodic wave solution and the kink (or shock) wave solution, if exist, to the equation under consideration in a unified way. The basic idea of this approach is as follows. For a given nonlinear evolution equation, say, in two variables

$$N(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{1.1}$$

we seek a travelling wave solution of the form

$$u(x, t) \equiv u(\xi), \qquad \xi = (x - ct). \tag{1.2}$$

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Without loss of generality, we can define $k > 0$. Substituting Eq. (1.2) into Eq. (1.1) yields an ordinary differential equation of $u(\xi)$. Then $u(\xi)$ is expanded into a polynomial in $f(\xi)$

$$u(\xi) = \sum_{i=0}^{n} A_i f^i, \tag{1.3}$$

where A_i are constants to be determined, n fixed by balancing the linear term of highest order with nonlinear term in Eq. (1.1), and f satisfies the following equation (the first kind of elliptic equation)

$$f'' = pf + qf^3, \quad f'^2 = pf^2 + \frac{1}{2}qf^4 + r, \tag{1.4}$$

where p, q, and r are constants to be determined. After Eq. (1.3) is substituted into the ordinary differential equation, the coefficients A_i, k, c, p, q, and r may be determined. If any of the parameters is left unspecified, it is regarded as being arbitrary for the solution to Eq. (1.1). Thus Eq. (1.3) establishes an algebraic mapping relation between the solution to Eq. (1.4) and that of Eq. (1.1). Because of the entrance of three parameters p, q, and r, Eq. (1.4) has rich structures of solutions. As $p = -2, q = 2$ and $r = 1$, for example, the solution of Eq. (1.4) reads $f(\xi) = \tanh\xi$, and the method is called tanh-function method. When $p = 1, q = -2$, and $r = 0$, Eq. (1.4) has solution $f(\xi) = \sec \mathrm{h}\xi$, and the method is named sech-function method. Above all, Eq. (1.4) has many Jacobi elliptic function solutions for different values of p, q, and r. So it may be said that the mapping method is a unified approach, including tanh-, sech-, and Jacobi elliptic function methods as special cases. In this paper, we further develop and modify this method to obtain new exact solutions to the combined KdV and mKdV equation. In Eq. (1.3), $u(\xi)$ is expanded into a polynomial in f with positive powers. Now we assume $u(\xi)$ may be expanded into a polynomial in f with both positive and negative powers, i.e., we take

$$u(\xi) = \sum_{i=0}^{n} A_i f^i + \sum_{i=1}^{n} B_i f^{-i}, \tag{1.5}$$

where n is the same as in Eq. (1.3), and f satisfies Eq. (1.4). When $B_i = 0$, Eq. (1.5) degenerates as Eq. (1.3). The other procedure is the same as the above.

2. MAIN RESULTS

The combined KdV and mKdV equation

$$u_t + \gamma u u_x + \alpha u^2 u_x + \beta u_{xxx} = 0, \tag{2.1}$$

represents a model for wave propagation in a one-dimensional nonlinear lattice, with anharmonic forces binding the particles (Wadati, 1975a,b). Various forms of this equation have been used in plasma physics, solid-state physics and quantum field theory (Dey, 1986; Konno and Ichikawa, 1974; Narayanamurti and Varma, 1970; Tappert and Varma, 1970). Many authors have obtained some exact solutions to Eq. (2.1) (Coffey, 1990; Mohamad, 1992; Yu, 2000). But our interest is confined to the determination of *new* exact solutions of it. Substituting Eq. (1.2) into Eq. (2.1) and integrating once, we find

$$-cu + \frac{1}{2}\gamma u^2 + \frac{1}{3}\alpha u^3 + \beta k^2 u'' = C, \tag{2.2}$$

where C is an integral constant. The substitution of Eq. (1.5) with $n = 1$ into Eq. (2.2) and use of Eq. (1.4) yields (equating the coefficients of like powers of f)

$$f^3 : \tfrac{1}{3}\alpha A_1^3 + q\beta k^2 A_1 = 0, \tag{2.3}$$

$$f^2 : \alpha A_0 A_1^2 + \tfrac{1}{2}\gamma A_1^2 = 0, \tag{2.4}$$

$$f^1 : -cA_1 + \alpha\left(A_0^2 A_1 + A_1^2 B_1\right) + p\beta k^2 A_1 + \gamma A_0 A_1 = 0, \tag{2.5}$$

$$f^0 : -cA_0 + \tfrac{1}{3}\alpha\left(A_0^3 + 6A_0 A_1 B_1\right) + \tfrac{1}{2}\gamma(A_0^2 + 2A_1 B_1) = C, \tag{2.6}$$

$$f^{-1} : -cB_1 + \alpha\left(A_0^2 B_1 + A_1 B_1^2\right) + p\beta k^2 B_1 + \gamma A_0 B_1 = 0, \tag{2.7}$$

$$f^{-2} : \alpha A_0 B_1^2 + \tfrac{1}{2}\gamma B_1^2 = 0, \tag{2.8}$$

$$f^{-3} : \tfrac{1}{3}\alpha B_1^3 + 2r\beta k^2 B_1 = 0, \tag{2.9}$$

from which it is found that

$$A_0 = -\frac{\gamma}{2\alpha}, \quad A_1 = \pm\sqrt{\frac{-3q\beta}{\alpha}}k, \quad B_1 = \pm\sqrt{\frac{-6r\beta}{\alpha}k,} \tag{2.10}$$

$$c = -\frac{\gamma^2}{4\alpha^2} \pm 3\beta k\sqrt{2qr} + p\beta k^2. \tag{2.11}$$

In Eq.(2.11), c takes positive sign when A_1 and B_1 assume the same sign, otherwise, c negative sign. Thus we obtain exact solution to Eq. (2.1)

$$u(x, t) = -\frac{\gamma}{2\alpha} \pm \sqrt{\frac{-3q\beta}{\alpha}}kf \pm \sqrt{\frac{-6r\beta}{\alpha}}kf^{-1}, \tag{2.12}$$

where f satisfies Eq. (1.4) and $\xi = k(x - ct)$ with c given by Eq. (2.11). As example, we only discuss specific expressions of $u(x, t)$ when positive sign is taken in Eqs. (2.10) and (2.11) for simplicity.

Case 2.1. $p = -2, \quad q = 2, r = 1.$

In this case, Eq. (1.4) has solution $f(\xi) = \tanh\xi$. So we obtain exact solution to Eq. (2.1)

$$u(x, t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{-6\beta}{\alpha}}k(\tanh\xi + \cos\mathrm{h}\xi csch\xi), \tag{2.13}$$

where $\xi = k(x - ct)$ with $c = -\frac{\gamma^2}{4\alpha^2} + 6\beta k - 2\beta k^2$, and which demands $\alpha > 0$, $\beta < 0$ or $\alpha < 0$, $\beta > 0$.

Case 2.2. $p = -(1 + m^2)$, $q = 2m^2$, $r = 1$.

The solution of Eq.(1.4) is $f(\xi) = sn\xi$ or $f(\xi) = cd\xi \equiv cn\xi/dn\xi$. Thus we get

$$u(x,t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{-6\beta}{\alpha}}k(msn\xi + sn^{-1}\xi), \tag{2.14}$$

and

$$u(x,t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{-6\beta}{\alpha}}k(mcd\xi + cd^{-1}\xi), \tag{2.15}$$

where $\xi = k(x - ct)$ with $c = -\frac{\gamma^2}{4\alpha^2} + 6\beta km - (1 + m^2)\beta k^2$, and which demand $\alpha > 0$, $\beta < 0$, or $\alpha < 0$, $\beta > 0$. As $m \to 1$, $sn\xi \to \tanh\xi$, and Eq.(2.14) degenerates as Eq.(2.13).

Case 2.3. $p = 2 - m^2$, $q = -2$, $r = -m'^2 \equiv -(1 - m^2)$.

The solution of Eq.(1.4) reads $f(\xi) = dn\xi$, and we obtain exact solution to Eq. (2.1)

$$u(x,t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{6\beta}{\alpha}}k(dn\xi + \sqrt{1 - m^2}\, dn^{-1}\xi), \tag{2.16}$$

where $\xi = k(x - ct)$ with $c = -\frac{\gamma^2}{4\alpha^2} + 6\beta k\sqrt{1 - m^2}(2 - m^2)\beta k^2$, and which demands $\alpha > 0$, $\beta > 0$, or $\alpha < 0$, $\beta < 0$.

Case 2.4. $p = 2 - m^2$, $q = 2$, $r = m'^2$.

Eq.(1.4) has solution $f(\xi) = cs\xi \equiv cn\xi/sn\xi$. The exact solution to Eq.(2.1) reads

$$u(x,t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{-6\beta}{\alpha}}k(cs\xi + \sqrt{1 - m^2}cs^{-1}\xi), \tag{2.17}$$

where $\xi = k(x - ct)$ with $c = -\frac{\gamma^2}{4\alpha^2} + 6\beta k\sqrt{1 - m^2} + (2 - m^2)\beta k^2$, and which demands $\alpha > 0$, $\beta < 0$ or $\alpha < 0$, $\beta > 0$. As $m \to 0$, $cs\xi \to \cot\xi$, and from Eq.(2.17) we obtain

$$u(x,t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{-6\beta}{\alpha}}k \sec\xi csc\xi. \tag{2.18}$$

Case 2.5. $p = -(1 + m^2)$, $q = 2$, $r = m^2$.

The solution to Eq.(1.4) is $f(\xi) = ns\xi \equiv 1/sn\xi$ or $f(\xi) = dc\xi \equiv dn\xi/cn\xi$. So we have Eq.(2.14) and

$$u(x, t) = -\frac{\gamma}{2\alpha} + \sqrt{\frac{-6\beta}{\alpha}}k(dc\xi + m dc^{-1}\xi), \tag{2.19}$$

where $\xi = k(x - ct)$ with $c = -\frac{\gamma^2}{4\alpha^2} + 6\beta km - (1 + m^2)\beta k^2$, and which demands $\alpha > 0$, $\beta < 0$ or $\alpha < 0$, $\beta > 0$. In equations above, $sn\xi$, $cn\xi$, and $dn\xi$ are Jacobi elliptic sine, cosine functions, and the third kind of Jacobi elliptic function, respectively. And $m(0 < m < 1)$ is the modulus of the elliptic function. Detailed explanations about Jacobi elliptic functions can be found in references (Bowman, 1959; Liu and Liu, 2000; Prasolov and Solovyev, 1997).

3. CONCLUSION

New exact solutions to the combined KdV and mKdV equation are obtained by means of the modified mapping method. It can be seen that because of the entrance of three parameters p, q, and r, we may obtain multiple exact solutions to the equation in question in a unified way, and only minimal algebra is needed to find these solutions. The method used in this paper is applicable to a large variety of nonlinear evolution equations, as long as even- and odd-order derivative terms do not coexist in the equation. It is interesting to extend further this approach to deal with nonlinear evolution equation with variable coefficients.

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